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CYLINDERS IN SINGULAR DEL PEZZO SURFACES

IVAN CHELTISOV, JIHUN PARK AND JOONYEONG WON

ABSTRACT. For each del Pezzo surface S with du Val singularities, we determine whether it admits a $(-K_S)$ -polar cylinder or not. If it allows one, then we present an effective \mathbb{Q} -divisor D that is \mathbb{Q} -linearly equivalent to $-K_S$ and such that the open set $S \setminus \text{Supp}(D)$ is a cylinder. As a corollary, we classify all the del Pezzo surfaces with du Val singularities that admit nontrivial \mathbb{G}_a -actions on their affine cones defined by their anticanonical divisors.

All considered varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic 0 throughout this article.

1. INTRODUCTION

Let X be a projective variety and H be an ample divisor on X . The *generalised cone* over the polarised variety (X, H) is the affine variety defined by

$$\hat{X} = \text{Spec} \left(\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nH)) \right).$$

The affine variety \hat{X} is the usual cone over the embedded image of X in a projective space by the linear system $|H|$ if H is very ample and the image of the variety X is projectively normal.

The question of whether the generalised cone of a given polarised variety (X, H) admits a nontrivial \mathbb{G}_a -action has been studied extensively in [6], [13], [14], [15] and [16]. The present article is focused on singular del Pezzo surfaces S_d polarised by anticanonical divisors $-K_{S_d}$ to extend the results in [6], [13] and [16] to the singular del Pezzo surfaces. Indeed, it classifies all the del Pezzo surfaces with du Val singularities that admit nontrivial \mathbb{G}_a -actions on their generalised cones over $(S_d, -K_{S_d})$.

Let S_d be a del Pezzo surface of degree d with at worst du Val singularities and let \hat{S}_d be the generalised cone over $(S_d, -K_{S_d})$. For $3 \leq d \leq 9$, the anticanonical divisor is very ample and the anticanonical linear system embeds S_d into the projective space \mathbb{P}^d . The embedded surface $S_d \subset \mathbb{P}^d$ is projectively normal. Therefore the generalised cone \hat{S}_d is the affine cone in \mathbb{A}^{d+1} over the variety embedded in \mathbb{P}^d . In particular, for $d = 3$, the surface S_3 anticanonically embedded in \mathbb{P}^3 is defined by a cubic homogenous polynomial equation, and hence the generalised cone \hat{S}_3 is the affine hypersurface in \mathbb{A}^4 defined by the same cubic polynomial equation. Meanwhile, for $d = 2$ (resp. $d = 1$), the generalised cone \hat{S}_d is the affine cone in \mathbb{A}^4 over the hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ (resp. $\mathbb{P}(1, 1, 2, 3)$) defined by a quasi-homogeneous polynomial of degree 4 (resp. 6) ([11, Theorem 4.4]).

The group of T. Kishimoto, Yu. Prokhorov, M. Zaidenberg and the group of I. Cheltsov, J. Park, J. Won have studied existence of nontrivial \mathbb{G}_a -actions on such affine cones and obtained results for smooth del Pezzo surfaces.

Theorem 1.1. *For a smooth del Pezzo surface S_d of degree $4 \leq d \leq 9$, its generalised cone \hat{S}_d admits an effective \mathbb{G}_a -action.*

Proof. See [13, Theorem 3.19]. □

Theorem 1.2. *For a smooth del Pezzo surface S_d of degree $d \leq 3$, its generalised cone \hat{S}_d admits no nontrivial \mathbb{G}_a -action.*

Proof. See [16, Theorem 1.1] and [6, Corollary 1.8]. \square

Their proofs make good use of a geometric property called cylindricity, which is worthwhile to study for its own sake.

Definition 1.3. Let M be a \mathbb{Q} -divisor on a projective normal variety X . An M -polar cylinder in X is an open subset

$$U = X \setminus \text{Supp}(D)$$

defined by an effective \mathbb{Q} -divisor D in the \mathbb{Q} -linear equivalence class of M such that U is isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z .

It is shown that the existence of a nontrivial \mathbb{G}_a -action on the generalised cone over (X, H) is equivalent to the existence of an H -polar cylinder on X .

Lemma 1.4. *Let X be a projective normal variety equipped with an ample Cartier divisor H on X . Suppose that the generalised cone \hat{X} over (X, H) is normal. Then \hat{X} admits an effective \mathbb{G}_a -action if and only if X contains an H -polar cylinder.*

Proof. See [16, Theorem 2.1]. \square

Indeed, in order to prove Theorems 1.1 and 1.2, they show that a smooth del Pezzo surface S_d has a $(-K_{S_d})$ -polar cylinder if $4 \leq d \leq 9$ but no $(-K_{S_d})$ -polar cylinder if $d \leq 3$.

The goal of the present article is to extend the results of [6], [13] and [16] to del Pezzo surfaces with du Val singularities. To be precise, we prove the following:

Theorem 1.5 (cf. Remark 3.8). *Let S_d be a del Pezzo surface of degree d with at most du Val singularities.*

- I. *The surface S_d does not admit a $(-K_{S_d})$ -polar cylinder when*
 - (1) $d = 1$ and S_d allows only singular points of types A_1, A_2, A_3, D_4 if any;
 - (2) $d = 2$ and S_d allows only singular points of type A_1 if any;
 - (3) $d = 3$ and S_d allows no singular point.
- II. *The surface S_d has a $(-K_{S_d})$ -polar cylinder if it is not one of the del Pezzo surfaces listed in I.*

Theorem 1.5 immediately implies the following via Lemma 1.4

Corollary 1.6. *Let S_d be a del Pezzo surface of degree d with at most du Val singularities. Then the affine cone over $(S_d, -K_{S_d})$ does not admit a nontrivial \mathbb{G}_a -action exactly when*

- (1) $d = 1$ and S_d allows only singular points of types A_1, A_2, A_3, D_4 if any;
- (2) $d = 2$ and S_d allows only singular points of type A_1 if any;
- (3) $d = 3$ and S_d allows no singular point,

As mentioned before, the theorem has been verified for smooth del Pezzo surfaces in [6], [13] and [16]. For this reason, from now on, we consider only singular del Pezzo surfaces. The cone over an irreducible conic curve in \mathbb{P}^3 obviously has a cylinder. Indeed, a quadruple ruling line is \mathbb{Q} -linearly equivalent to the anticanonical class and its complement is isomorphic to \mathbb{A}^2 . Therefore, we may exclude this singular del Pezzo surface from our consideration.

2. PRELIMINARIES

2.1. Singularities and Inequalities. Let S be a projective surface with at most du Val singularities. In addition, let D be an effective \mathbb{Q} -divisor on S .

Lemma 2.1. *If the log pair (S, D) is not log canonical at a smooth point P , then $\text{mult}_P(D) > 1$.*

Proof. See [19, Proposition 9.5.13]. \square

Write $D = \sum_{i=1}^r a_i D_i$, where D_i 's are distinct prime divisors and a_i 's are positive rational numbers.

Lemma 2.2. *Let T be an effective \mathbb{Q} -divisor on S other than the divisor D such that $T \sim_{\mathbb{Q}} D$ and $\text{Supp}(T) \subset \text{Supp}(D)$. For every non-negative rational number ϵ , put $D_\epsilon = (1 + \epsilon)D - \epsilon T$. Then*

- (1) $D_\epsilon \sim_{\mathbb{Q}} D$ for every ϵ ;
- (2) the set $\{\epsilon \in \mathbb{Q}_{>0} \mid D_\epsilon \text{ is effective}\}$ attains the maximum μ ;
- (3) at least one component of the support of the divisor T is not contained in the support of the divisor D_μ ;
- (4) if the log pair (S, T) is log canonical at a point P but (S, D) is not log canonical at P , then the log pair (S, D_μ) is not log canonical at P .

Proof. See [6, Lemma 2.2]. □

The following is a ready-made adjunction for our situation.

Lemma 2.3. *Suppose that the log pair (S, D) is not log canonical at a smooth point P . If a component D_j with $a_j \leq 1$ is smooth at P , then*

$$D_j \cdot \left(\sum_{i \neq j} a_i D_i \right) \geq \sum_{i \neq j} a_i (D_i \cdot D_j)_P > 1,$$

where $(D_i \cdot D_j)_P$ is the local intersection number of D_i and D_j at P .

Proof. See [18, Theorem 5.50]. □

The following is an easy application of Lemma 2.3.

Lemma 2.4. *Suppose that the surface S has a singular point P of type D_4 . Let $g: \tilde{S} \rightarrow S$ be the minimal resolution of the point P . Denote by E_1, E_2, E_3 and E_4 the g -exceptional curves, where E_3 is the (-2) -curve intersecting the other three (-2) -curves. Write*

$$\tilde{D} = g^*(D) - \sum_{i=1}^4 a_i E_i,$$

where \tilde{D} is the proper transform of D by g . Then the log pair (S, D) is not log canonical at P if and only if $a_3 > 1$.

Proof. See [5, Lemma 2.5]. □

Remark 2.5. Let $f: \bar{S} \rightarrow S$ be the blow up of the surface S at a smooth point P with the exceptional divisor E and let \bar{D} be the proper transform of D by f . Then we have

$$K_{\bar{S}} + \bar{D} + (\text{mult}_P(D) - 1)E = f^*(K_S + D).$$

The log pair (S, D) is log canonical at P if and only if the log pair $(\bar{S}, \bar{D} + (\text{mult}_P(D) - 1)E)$ is log canonical along E . If (S, D) is not log canonical at P , then there exists a point Q on E at which $(\bar{S}, \bar{D} + (\text{mult}_P(D) - 1)E)$ is not log canonical. Lemma 2.1 then implies

$$(2.6) \quad \text{mult}_P(D) + \text{mult}_Q(\bar{D}) > 2.$$

If $\text{mult}_P(D) \leq 2$, then $(\bar{S}, \bar{D} + (\text{mult}_P(D) - 1)E)$ is log canonical at every point on E except the point Q . Indeed, if it is not log canonical at another point O on E , then Lemma 2.3 yields a contradiction,

$$2 \geq \text{mult}_P(D) = \bar{D} \cdot E \geq \text{mult}_Q(\bar{D}) + \text{mult}_O(\bar{D}) > 2.$$

The following lemma will be useful for the article.

Lemma 2.7. *Let C_1 and C_2 be irreducible curves on the surface S that both are smooth at a smooth point P and intersect transversally at P . In addition, let Ω be an effective \mathbb{Q} -divisor on S whose support contains neither C_1 nor C_2 . Suppose that the log pair $(S, a_1C_1 + a_2C_2 + \Omega)$ is not log canonical at P for some non-negative rational numbers a_1, a_2 . If $\text{mult}_P(\Omega) \leq 1$, then either*

$$\text{mult}_P(\Omega \cdot C_1) > 2(1 - a_2) \quad \text{or} \quad \text{mult}_P(\Omega \cdot C_2) > 2(1 - a_1).$$

Proof. See [4, Theorem 13]. □

From now on, on a projective surface, an effective \mathbb{Q} -divisor \mathbb{Q} -linearly equivalent to the anticanonical class of the surface will be called *an effective anticanonical \mathbb{Q} -divisor* and a member of the anticanonical linear system will be called *an effective anticanonical divisor*.

2.2. Singularity types. For singular del Pezzo surfaces S of degrees 1 and 2 not listed in Theorem 1.5 (I), the construction method of their $(-K_S)$ -polar cylinders will be given according to the singularity types of the surfaces S . For this purpose, we adopt the following definition.

Definition 2.8. Let S_1 and S_2 be del Pezzo surfaces with at most du Val singularities and let \tilde{S}_1 and \tilde{S}_2 be their minimal resolutions, respectively. We say that the del Pezzo surfaces S_1 and S_2 have the same singularity type (or the minimal resolutions \tilde{S}_1 and \tilde{S}_2 have the same type) if there is an isomorphism of the Picard group of \tilde{S}_1 to the Picard group of \tilde{S}_2 preserving the intersection form that gives a bijection between their sets of classes of negative curves.

Note that the minimal resolutions of del Pezzo surfaces with du Val singularities are smooth weak del Pezzo surfaces, i.e., smooth projective surfaces with nef and big anticanonical divisors.

It is known that the types of smooth weak del Pezzo surfaces of degree d are in one-to-one correspondence to the subsystems of the root systems of types $E_8, E_7, E_6, D_5, A_4, A_2 + A_1, A_1$, respectively, for $d = 1, \dots, 7$, with four exceptions: $8A_1, 7A_1, D_4 + 4A_1$ for $d = 1$ and $7A_1$ for $d = 2$ (see [1], [2], [7], [9], [24], [26]).

Since the isomorphisms of the Picard groups of the weak del Pezzo surfaces of the same type preserve the intersection forms, we can conclude from [8, Théorème III.2 and Corollaire] that a given singularity type has a unique configuration of (-1) -curves and (-2) -curves. The type of smooth weak del Pezzo surface is uniquely determined by its degree and its configuration of (-1) -curves and (-2) -curves. Consequently, for a given singularity type of del Pezzo surfaces of degree d , if we find one weak del Pezzo surface of degree d whose corresponding singular del Pezzo surface has the given singularity type, then this weak del Pezzo surface gives us the configuration of (-1) -curves and (-2) -curves for the given singularity type since every del Pezzo surface of the same singularity type has the same configuration of (-1) -curves and (-2) -curves on its weak del Pezzo surface.

For the singularity types of del Pezzo surfaces of degrees ≤ 2 with at most du Val singularities, we refer to the table in [26]. The table completely classifies subsystems of the root systems E_7 and E_8 up to actions of their Weyl groups.

On a del Pezzo surface of a given degree d , the configuration of the (-2) -curves on the corresponding smooth weak del Pezzo surface does not determine the type uniquely. In such a case, there are precisely two types. The following are the ADE-types (with the degrees $d \leq 2$) that have two different singularity types.

$$\begin{aligned} d = 1. & \quad A_7, A_5 + A_1, 2A_3, A_3 + 2A_1, 4A_1; \\ d = 2. & \quad A_5 + A_1, A_5, A_3 + 2A_1, A_3 + A_1, 4A_1, 3A_1; \end{aligned}$$

We need to distinguish these singularity types of del Pezzo surfaces of degrees $d \leq 2$ with the same ADE-types. However, we do not have to consider the ADE-types $2A_3, A_3 + 2A_1, 4A_1$ for $d = 1$ and $4A_1, 3A_1$ for $d = 2$ due to Theorem 1.5 (I). For the remaining ADE-types, the vertex v in the Dynkin diagram of A_{2n+1} , $n \geq 1$, is called the central vertex if $A_{2n+1} - v = 2A_n$.

For the ADE-types A_7 ($d = 1$), A_5 ($d = 2$), $A_5 + A_1$ ($d = 2$), A_3 ($d = 4$), we use $(A_7)'$, $(A_5)'$, $(A_5 + A_1)'$ and $(A_3)'$ if there are (-1) -curves intersecting the (-2) -curves corresponding to the central vertices and we use $(A_7)''$, $(A_5)''$, $(A_5 + A_1)''$ and $(A_3)''$ if there are not such (-1) -curves.

For the ADE-types $A_5 + A_1$ ($d = 1$), $A_3 + A_1$ ($d = 2$), $A_3 + 2A_1$ ($d = 2$), we use $(A_5 + A_1)'$, $(A_3 + A_1)'$, $(A_3 + 2A_1)'$ if there are (-1) -curves intersecting the (-2) -curves corresponding to the central vertices and the vertices of A_1 and we use $(A_5 + A_1)''$, $(A_3 + A_1)''$, $(A_3 + 2A_1)''$ if there are not such (-1) -curves.

3. ABSENCE OF CYLINDERS

3.1. Del Pezzo surfaces of degree 1. Let S be a del Pezzo surface of degree 1 with at most du Val singularities. Then its anticanonical linear system $|-K_S|$ is a pencil that has a unique base point. Denote its base point by O . Note that the base point O must be a smooth point of the surface S .

Theorem 3.1. *Let D be an effective anticanonical \mathbb{Q} -divisor on S .*

- (1) *The log pair (S, D) is log canonical outside of finitely many points.*
- (2) *It is log canonical at the point O .*

Let P be either a smooth point different from O or a singular point of type A_1 , A_2 , A_3 or D_4 and let C be the curve in the pencil $|-K_S|$ that passes through P .

- (3) *If the log pair (S, D) is not log canonical at P , then*
 - *the log pair (S, C) is not log canonical at P ;*
 - *the support of D contains the support of C .*

Proof. Since $-K_S$ is ample, the first statement immediately follows from $-K_S \cdot D = 1$

For a general member Z in the anticanonical linear system $|-K_S|$, we have

$$1 = Z \cdot D \geq \text{mult}_O(Z) \text{mult}_O(D) \geq \text{mult}_O(D).$$

It then follows from Lemma 2.1 that the log pair (S, D) is log canonical at the base point O .

Now we consider a point P on S other than O . For (3) we first prove that (S, D) is log canonical at P if the support of D does not contain the support of C . For this purpose, we suppose that the support of the curve C is not contained in the support of D . Note that C is irreducible.

If P is a smooth point, then we can obtain

$$1 = C \cdot D \geq \text{mult}_P(C) \text{mult}_P(D) \geq \text{mult}_P(D),$$

which implies that (S, D) is log canonical at P by Lemma 2.1.

Now we suppose that P is a singular point of the surface S . Let $f: \tilde{S} \rightarrow S$ be the minimal resolution of the singular point P . Denote by E_1, \dots, E_r the f -exceptional curves, denote by \tilde{D} the proper transform of the divisor D on the surface \tilde{S} and denote by \tilde{C} the proper transform of the curve C on the surface \tilde{S} . Then there are non-negative rational numbers a_1, \dots, a_r such that

$$K_{\tilde{S}} + \tilde{D} + \sum_{i=1}^r a_i E_i = f^*(K_S + D) \sim_{\mathbb{Q}} 0.$$

We can immediately see how the proper transform \tilde{C} of the effective anticanonical divisor C intersects the exceptional divisors E_i (for instance, see [21, Appendix]).

Suppose that P is a singular point of type D_4 . Then $r = 4$ and we may assume that the exceptional divisor E_3 is the (-2) -curve that intersects all the other three (-2) -curves. We see from [21, Appendix] that $\tilde{C} \cdot E_3 = 1$ and $\tilde{C} \cdot E_1 = \tilde{C} \cdot E_2 = \tilde{C} \cdot E_4 = 0$. We then obtain

$$1 - a_3 = \left(f^*(-K_S) - \sum_{i=1}^r a_i E_i \right) \cdot \tilde{C} = \tilde{D} \cdot \tilde{C} \geq 0.$$

Lemma 2.4 therefore implies that (S, D) is log canonical at P .

Suppose that P is a singular point of type A_r . We assume that E_1 and E_r are the tail curves, i.e., the (-2) -curves intersecting only one (-2) -curve, respectively. Then the curve \tilde{C} intersects E_1 and E_r , respectively, at one point transversally (if $r = 1$, then $\tilde{C} \cdot E_1 = 2$). But it does not intersect the other (-2) -curves. Therefore,

$$1 - a_1 - a_r = \left(f^*(-K_S) - \sum_{i=1}^r a_i E_i \right) \cdot \tilde{C} = \tilde{D} \cdot \tilde{C} \geq 0,$$

and hence $a_1 + a_r \leq 1$ (if $r = 1$, then $a_1 \leq \frac{1}{2}$).

Consider the case $r = 1$. Since $\tilde{D} \cdot E_1 = 2a_1 \leq 1$, the log pair $(\tilde{S}, \tilde{D} + a_1 E_1)$ is log canonical along the exceptional curve E_1 by Lemma 2.3. Therefore, (S, D) is log canonical at P .

Next we consider the case $r = 2$. We then have $a_1 + a_2 \leq 1$. Moreover, we obtain $2a_1 \geq a_2$ from the inequality

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq 0.$$

Similarly, $2a_2 \geq a_1$. Since $a_1 + a_2 \leq 1$, we may assume that $a_1 \leq \frac{1}{2}$. We obtain $(\tilde{D} + a_2 E_2) \cdot E_1 = 2a_1 \leq 1$, and hence $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2)$ is log canonical along the curve E_1 . Furthermore, the inequality

$$\tilde{D} \cdot E_2 = 2a_2 - a_1 \leq 2a_1 + (a_2 - a_1) = a_1 + a_2 \leq 1$$

implies that $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2)$ is log canonical along the curve E_2 . Consequently, (S, D) is log canonical at P .

Finally we consider the case $r = 3$. We have $a_1 + a_3 \leq 1$. Moreover, we may obtain $2a_1 \geq a_2$, $2a_2 \geq a_1 + a_3$ and $2a_3 \geq a_2$ from

$$\begin{cases} 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq 0, \\ 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq 0, \\ 2a_3 - a_2 = \tilde{D} \cdot E_3 \geq 0. \end{cases}$$

We may assume that $a_1 \leq \frac{1}{2}$ since $a_1 + a_3 \leq 1$. Since $(\tilde{D} + a_2 E_2 + a_3 E_3) \cdot E_1 = 2a_1 \leq 1$, the log pair $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is log canonical along the curve E_1 . In addition, $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is log canonical along $E_2 \setminus E_3$ and $E_3 \setminus E_2$ since

$$\begin{cases} \tilde{D} \cdot E_2 = 2a_2 - a_1 - a_3 \leq 2(a_1 + a_3) - (a_1 + a_3) = a_1 + a_3 \leq 1, \\ \tilde{D} \cdot E_3 = 2a_3 - a_2 \leq (2a_2 - a_1) + a_3 - a_2 \leq a_1 + a_3 \leq 1. \end{cases}$$

Let Q be the intersection point of E_2 and E_3 . We have

$$\begin{cases} \tilde{D} \cdot E_2 = 2a_2 - a_1 - a_3 \leq (4a_1 - a_1 + a_3) - 2a_3 = 2a_1 + (a_1 + a_3) - 2a_3 \leq 2(1 - a_3), \\ \tilde{D} \cdot E_3 = 2a_3 - a_2 = 2a_3 + a_2 - 2a_2 \leq 2a_3 + 2a_1 - 2a_2 \leq 2(1 - a_2). \end{cases}$$

Since $\text{mult}_Q(\tilde{D}) \leq \tilde{D} \cdot E_3 = 2a_3 - a_2 \leq 1$, Lemma 2.7 implies that $(\tilde{S}, \tilde{D} + a_2 E_2 + a_3 E_3)$ is log canonical at Q , and hence $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is log canonical at Q . Consequently, $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is log canonical along the three exceptional curves, and hence (S, D) is log canonical at P .

If the log pair (S, C) is log canonical at P , then we can obtain an effective anticanonical \mathbb{Q} -divisor D_μ from Lemma 2.2 such that (S, D_μ) is not log canonical at P and whose support does not contain the support of C . This however contradicts what we have proven so far. Therefore, (S, C) is not log canonical at P . \square

It is a common experience that D_4 -singularity is more singular than A_4 -singularity. However, to our surprise, Theorem 3.1 (3) does not hold for a singular point of type A_4 even though every singular point of type D_4 enjoys Theorem 3.1 (3).

3.2. Del Pezzo surfaces of degree 2. Let S be a del Pezzo surface of degree 2 with at most ordinary double points. Its anticanonical linear system $|-K_S|$ is base-point-free and induces a double cover $\pi: S \rightarrow \mathbb{P}^2$ ramified along a reduced quartic curve $R \subset \mathbb{P}^2$. Moreover, the curve R has at most ordinary double points. Note that the curve R may be reducible.

Let D be an effective anticanonical \mathbb{Q} -divisor on S .

Lemma 3.2. *Write*

$$D = \mu C + \Omega,$$

where μ is a non-negative rational number, C is an irreducible curve and Ω is an effective \mathbb{Q} -divisor whose support does not contain the curve C . If $\mu > 1$, then $-K_S \cdot C = 1$ and $\pi(C)$ is a line in \mathbb{P}^2 that is an irreducible component of R .

Proof. Since $-K_S$ is ample, $-K_S \cdot C$ is a positive integer. The equality $-K_S \cdot C = 1$ immediately follows from

$$2 = -K_S \cdot (\mu C + \Omega) = -\mu K_S \cdot C - K_S \cdot \Omega \geq -\mu K_S \cdot C > -K_S \cdot C.$$

This shows that $\pi(C)$ is a line in \mathbb{P}^2 .

Suppose that $\pi(C)$ is not an irreducible component of R . Then there exists a curve C' different from C such that $C + C' \sim -K_S$ and $\pi(C') = \pi(C)$. Write $\Omega = \mu' C' + \Delta$, where Δ is an effective \mathbb{Q} -divisor on S whose support does not contain the curve C' . Since $\mu > 1$, we obtain $\mu' < 1$. Therefore, by taking $\frac{1}{1-\mu'}(D - \mu'(C + C'))$ instead of D , we may assume that $\mu' = 0$.

Since the intersection number $C \cdot C'$ belongs to $\frac{1}{2}\mathbb{Z}$, from

$$1 = D \cdot C' = \mu C \cdot C' + \Omega \cdot C' > C \cdot C',$$

we conclude that $C \cdot C' = \frac{1}{2}$. Therefore, $C^2 = -K_S \cdot C - C \cdot C' = \frac{1}{2}$. This implies that C passes through three ordinary double points, which is impossible unless $\pi(C)$ is an irreducible component of R . \square

Theorem 3.3. *Let P be a smooth point of S .*

(1) *If $\pi(P) \notin R$, then (S, D) is log canonical at P .*

Suppose $\pi(P) \in R$ and let T_P be the unique divisor in $|-K_S|$ that is singular at P .

(2) *If the log pair (S, D) is not log canonical at P , then*

- *the log pair (S, T_P) is not log canonical at P ;*
- *the support of D contains the support of T_P .*

Proof. For (1), the proof of [6, Lemma 3.2] works verbatim even though we allow more than two ordinary double points.

For (2), we suppose $\pi(P) \in R$. If the divisor T_P is reduced, then the proofs of [6, Lemmas 3.4 and 3.5] verifies the statement.

If the divisor T_P is not reduced, then $T_P = 2C$ for some irreducible smooth curve C and $\pi(C)$ is a line in \mathbb{P}^2 that is an irreducible component of the quartic curve R . It is clear that (S, T_P) is not log canonical at P . If $C \not\subset \text{Supp}(D)$, then

$$1 = C \cdot D \geq \text{mult}_P(C) \text{mult}_P(D) \geq \text{mult}_P(D),$$

and hence (S, D) is log canonical at P by Lemma 2.1. Therefore, the support of D must contain the support of C . \square

Suppose that (S, D) is not log canonical at a singular point P of S . Let $f: \bar{S} \rightarrow S$ be the blow up of S at P . Denote by E the f -exceptional curve and denote by \bar{D} the proper transform of the divisor D on the surface \bar{S} . Then

$$\bar{D} = f^*(D) - aE \sim_{\mathbb{Q}} f^*(-K_S) - aE$$

for some positive rational number a . This gives

$$K_{\bar{S}} + \bar{D} + aE = f^*(K_S + D) \sim_{\mathbb{Q}} 0,$$

which implies that $(\bar{S}, \bar{D} + aE)$ is not log canonical at some point Q on E by Remark 2.5.

Let H be a general curve in $|-K_S|$ that passes through P . Denote by \bar{H} its proper transform on the surface \bar{S} . Then $\bar{H} \cdot E = 2$. We have

$$0 \leq \bar{H} \cdot \bar{D} = \bar{H} \cdot (f^*(-K_S) - aE) = 2 - 2a,$$

which gives $a \leq 1$. Now applying Lemma 2.3 to $(\bar{S}, aE + \bar{D})$ and E , we get

$$2a = E \cdot \bar{D} \geq \text{mult}_Q(E \cdot \bar{D}) > 1.$$

Consequently, we see $\frac{1}{2} < a \leq 1$. Since $a \leq 1$, the log pair $(\bar{S}, \bar{D} + aE)$ is log canonical at every point of E other than the point Q by Remark 2.5.

Since $-K_{\bar{S}} = f^*(-K_S)$, the linear system $|-K_{\bar{S}} - E|$ is a pencil. In fact, it is a base-point-free pencil. A general curve in $|-K_{\bar{S}} - E|$ is a smooth rational curve that intersects E by two distinct points. Moreover, since $|-K_{\bar{S}} - E|$ does not have any base points, there exists a unique curve $C \in |-K_S|$ whose proper transform \bar{C} by f passes through the point Q .

Theorem 3.4. *Let P be an ordinary double point of S and let C be the curve in $|-K_S|$ described above. If the log pair (S, D) is not log canonical at P , then*

- *the log pair (S, C) is not log canonical at P ;*
- *the support of D contains the support of C .*

Proof. We suppose that $\text{Supp}(D)$ does not contain the support of C and then look for a contradiction. We have three cases as below.

Case 1. The curve C is not reduced.

Then $C = 2L$, where L is a smooth rational curve on S such that $\pi(L)$ is a line in \mathbb{P}^2 and it is an irreducible component of the curve R .

Denote by \bar{L} the proper transform of the curve L on the surface \bar{S} . Then the point Q belongs to \bar{L} by the choice of C . Since $L \not\subset \text{Supp}(D)$, then

$$1 - a = \bar{L} \cdot \bar{D} \geq \text{mult}_Q(\bar{D}),$$

and hence that $1 \geq a + \text{mult}_Q(\bar{D}) = \text{mult}_Q(\bar{D} + aE)$. Therefore, $(\bar{S}, \bar{D} + aE)$ is log canonical at Q by Lemma 2.1. This is a contradiction.

Case 2. The curve C is reduced and irreducible.

Put $m = \text{mult}_Q(\bar{D})$. From

$$2 - 2a = \bar{C} \cdot \bar{D} \geq m,$$

we obtain $m + 2a \leq 2$. Note that $m \leq 2 - 2a < 1$ since $a > \frac{1}{2}$.

Let $g: \check{S} \rightarrow \bar{S}$ be the blow up of the surface \bar{S} at the point Q . Denote by F the g -exceptional curve and denote by \check{E} and \check{D} the proper transforms of the divisors E and \bar{D} on the surface \check{S} , respectively. Then

$$K_{\check{S}} + \check{D} + a\check{E} + (a + m - 1)F = g^*(K_{\bar{S}} + \bar{D} + aE),$$

and $(\check{S}, \check{D} + a\check{E} + (a + m - 1)F)$ is not log canonical at some point O of the exceptional curve F .

Since $a + m - 1 \leq 1$, the inequality

$$\text{mult}_O(\check{D}) \leq F \cdot \check{D} = m \leq 1$$

implies that $(\check{S}, \check{D} + (a + m - 1)F)$ is log canonical along the divisor F by Lemma 2.3. Therefore, the point O must be the intersection point of F and \check{E} .

Since $\text{mult}_O(\check{D}) \leq \text{mult}_Q(\bar{D}) = m \leq 1$, we can apply Lemma 2.7 to the log pair $(\check{S}, \check{D} + a\check{E} + (a + m - 1)F)$ at the point O , so that we obtain either

$$2a - m = \check{D} \cdot \check{E} > 2(2 - a - m) \quad \text{or} \quad m = \check{D} \cdot F > 2(1 - a).$$

However, both the inequalities are impossible since $m + 2a \leq 2$. This is a contradiction.

Case 3. The curve C is reduced but reducible.

The curve C consists of two distinct smooth irreducible and reduced curves L_1 and L_2 . Note that $-K_S \cdot L_1 = -K_S \cdot L_2 = 1$ and these two curves intersect at the point P . Without loss of generality, we may assume that the curve L_1 is not contained in the support of D . Then we put $D = bL_2 + \Omega$, where b is a non-negative rational number and Ω is an effective \mathbb{Q} -divisor on S whose support does not contain the curve L_2 . Denote by \bar{L}_1 , \bar{L}_2 and $\bar{\Omega}$ the proper transforms of the curves L_1 , L_2 and the divisor Ω on the surface \bar{S} , respectively. Note that $\bar{L}_1 \cdot E = \bar{L}_2 \cdot E = 1$.

The point Q cannot belong to the curve \bar{L}_1 . Indeed, if so, then

$$1 - a = \bar{L}_1 \cdot \bar{D} \geq \text{mult}_Q(\bar{D}),$$

and hence $\text{mult}_Q(\bar{D} + aE) \leq 1$. Since $(\bar{S}, \bar{D} + aE)$ is not log canonical at Q , this is absurd. Therefore, the point Q must belong to the curve \bar{L}_2 .

Recall that $\pi(L_1) = \pi(L_2)$ is a line in \mathbb{P}^2 that passes through the point $\pi(P)$. Since $Q \notin \bar{L}_1$ and $Q \in \bar{L}_2$, the intersection $L_1 \cap L_2$ consists of two distinct points, one of which is the point P . Thus, the intersection $\bar{L}_1 \cap \bar{L}_2$ consists of a single point. This point can be either a smooth point or an ordinary double point of the surface \bar{S} . In the former case, we have $\bar{L}_1 \cdot \bar{L}_2 = 1$ and $\bar{L}_1^2 = \bar{L}_2^2 = -1$. In the latter case, we have $\bar{L}_1 \cdot \bar{L}_2 = \frac{1}{2}$ and $\bar{L}_1^2 = \bar{L}_2^2 = -\frac{1}{2}$.

From

$$1 - a - b(\bar{L}_1 \cdot \bar{L}_2) = \bar{\Omega} \cdot \bar{L}_1 \geq 0$$

we obtain $a + b(\bar{L}_1 \cdot \bar{L}_2) \leq 1$. Therefore,

$$\bar{L}_2 \cdot \bar{\Omega} = 1 - a - b\bar{L}_2^2 \leq (1 - a) \left(1 - \frac{\bar{L}_2^2}{\bar{L}_1 \cdot \bar{L}_2} \right) = 2(1 - a)$$

and

$$E \cdot \bar{\Omega} = 2a - b \leq 2 - b(2\bar{L}_1 \cdot \bar{L}_2 + 1) \leq 2(1 - b).$$

Meanwhile, from

$$\begin{cases} 1 - a - b\bar{L}_2^2 = \bar{L}_2 \cdot \bar{\Omega} \geq \text{mult}_Q(\bar{\Omega}), \\ 2a - b = E \cdot \bar{\Omega} \geq \text{mult}_Q(\bar{\Omega}), \end{cases}$$

we obtain $2\text{mult}_Q(\bar{\Omega}) \leq 1 + a - (1 + \bar{L}_2^2)b \leq 1 + a$. Therefore, $\text{mult}_Q(\bar{\Omega}) \leq 1$. This enables us to apply Lemma 2.7 to $(\bar{S}, \bar{\Omega} + aE + b\bar{L}_2)$ at the point Q . The log pair $(\bar{S}, \bar{\Omega} + aE + b\bar{L}_2)$ must be log canonical at Q . This is a contradiction.

These three cases lead to the conclusion that the support of D contain the curve C if (S, D) is not log canonical at P .

In Case 1, it is obvious that (S, C) is not log canonical at P . In Cases 2 and 3, if (S, C) is log canonical at P , then we can obtain an effective anticanonical \mathbb{Q} -divisor D_μ from Lemma 2.2 such that (S, D_μ) is not log canonical at P and whose support does not contain the support of C . This however contradicts what we have proven in Cases 2 and 3. Therefore, (S, C) is not log canonical at P . \square

3.3. Proof of Theorem 1.5 (I). Now we are ready to prove Theorem 1.5 (I), i.e., if a surface S is either a del Pezzo surface of degree 2 with only ordinary double points or a del Pezzo surface of degree 1 with du Val singularities of types A_1, A_2, A_3, D_4 only, then it cannot admit any $(-K_S)$ -polar cylinder.

To this end, we suppose that the del Pezzo surface S contains a $(-K_S)$ -polar cylinder and then we look for a contradiction.

Since S contains a $(-K_S)$ -polar cylinder, there is an effective anticanonical \mathbb{Q} -divisor D such that $U = S \setminus \text{Supp}(D)$ is isomorphic to an affine variety $Z \times \mathbb{A}^1$ for some smooth rational affine curve Z . Put $D = \sum_{i=1}^r a_i D_i$, where each D_i is an irreducible and reduced curve and each a_i is a positive rational number.

Lemma 3.5. *The components of D generate the divisor class group $\text{Cl}(S)$ of the surface S . In particular, the number of the irreducible components of D is at least the rank of $\text{Cl}(S)$.*

Proof. See [16, Lemma 4.6]. □

The natural projection $U \cong Z \times \mathbb{A}^1 \rightarrow Z$ induces a rational map $\phi: S \dashrightarrow \mathbb{P}^1$. Denote by \mathcal{L} the pencil on the surface S that induces the rational map ϕ . Then either the pencil \mathcal{L} is base-point-free or its base locus consists of a single point.

Lemma 3.6. *The pencil \mathcal{L} is not base-point-free.*

Proof. Suppose that the pencil \mathcal{L} is base-point-free. Then ϕ is a morphism, which implies that there exists exactly one irreducible component of $\text{Supp}(D)$ that does not lie in the fibers of ϕ . Moreover, this irreducible component is a section. Without loss of generality, we may assume that this component is D_r . Let L be a sufficiently general curve in \mathcal{L} . Then

$$2 = -K_S \cdot L = D \cdot L = \sum_{i=1}^r a_i D_i \cdot L = a_r D_r \cdot L,$$

and hence $a_r = 2$.

By Theorem 3.1 (1), the surface S cannot be of degree 1, and hence it must be of degree 2. Then the anticanonical linear system $| -K_S |$ is base-point-free and induces a double cover $\pi: S \rightarrow \mathbb{P}^2$ ramified along a reduced quartic curve $R \subset \mathbb{P}^2$. Moreover, the curve R has at most ordinary double points. Furthermore, it follows from Lemma 3.2 that $\pi(D_r)$ is a line in \mathbb{P}^2 that is an irreducible component of R . Therefore, $-K_S \sim 2D_r$, and hence $r = 1$. This implies that the rank of the divisor class group of S is one by Lemma 3.5. However, since the curve R has at most six singular points, the surface S can attain at most six ordinary double points. Therefore, the rank of the divisor class group of S is at least two. This is a contradiction. □

Denote the unique base point of the pencil \mathcal{L} by P . Resolving the base locus of the pencil \mathcal{L} we obtain a commutative diagram

$$\begin{array}{ccc} & W & \\ f_1 \swarrow & & \searrow f_2 \\ S & \xrightarrow{\phi} & \mathbb{P}^1, \end{array}$$

where f_1 is a composition of blow ups at smooth points over the point P and f_2 is a morphism whose general fiber is a smooth rational curve. Denote by E_1, \dots, E_n the exceptional curves of the birational morphism f_1 . Then there exists exactly one curve among them that does not lie in the fibers of the morphism f_2 . Without loss of generality, we may assume that this curve is E_n . The curve E_n is a section of the morphism f_2 .

For every D_i , denote by \hat{D}_i its proper transform on the surface W . Every curve \hat{D}_i lies in a fiber of the morphism f_2 .

Lemma 3.7. *Let D' be an effective anticanonical \mathbb{Q} -divisor on S with $\text{Supp}(D') \subseteq \text{Supp}(D)$. Denote by \hat{D}' its proper transform on W . Then*

$$K_W + \hat{D}' = f_1^*(K_S + D') + \sum_{i=1}^n c_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^n c_i E_i$$

for some rational numbers c_1, \dots, c_n . Moreover, we have $c_n = -2$. In particular, the log pair (S, D') is not log canonical at the point P .

Proof. The existence of rational numbers c_1, \dots, c_n is obvious. We must show that $c_n = -2$. Write $D' = \sum_{i=1}^r b_i D_i$, where each b_i is a non-negative rational number. Let L be a sufficiently general fiber of the morphism f_2 . Then

$$-2 = K_W \cdot L = K_W \cdot L + \sum_{i=1}^r b_i \hat{D}_i \cdot L = \sum_{i=1}^n c_i E_i \cdot L = c_n,$$

because E_n is a section of the morphism f_2 , every curve \hat{D}_i lies in a fiber of the morphism f_2 and every curve E_i with $i < n$ also lies in a fiber of the morphism f_2 . Hence, $c_n = -2$ and (S, D') is not log canonical at P . \square

Lemma 3.7 shows that if a del Pezzo surface S' with at worst du Val singularities contains a $(-K_{S'})$ -polar cylinder, then the surface S' must possess an effective anticanonical \mathbb{Q} -divisor B such that the log pair (S', B) is not log canonical. Such an effective anticanonical \mathbb{Q} -divisor is called a *tiger* on the del Pezzo surface S' (cf. [12]). In particular, applying Lemma 3.7 to (S, D) , we see that (S, D) is not log canonical at P .

Proof of Theorem 1.5 (I).

Case 1. The surface S is of degree 1.

By Theorem 3.1 (2), P is not the base point of the pencil $|-K_S|$. Thus, there exists a unique curve C in the pencil $|-K_S|$ that passes through P . If the rank of the divisor class group of S is greater than one, then $D \neq C$ by Lemma 3.5. If the rank is one, then the open set $S \setminus C$ must contain a singular point. But $S \setminus \text{Supp}(D) \cong \mathbb{A}^1 \times Z$ is smooth. Therefore, $D \neq C$.

Let μ be the greatest rational number such that $D' = (1 + \mu)D - \mu C$ is effective. Then (S, D') is not log canonical at P by Lemma 3.7. This contradicts Theorem 3.1 (3).

Case 2. The surface S is of degree 2.

We have the double cover $\pi: S \rightarrow \mathbb{P}^2$ ramified along a reduced quartic curve $R \subset \mathbb{P}^2$ given by the anticanonical linear system.

The surface S has at most six ordinary double points. If it has six ordinary double points, then the quartic curve R consists of four distinct lines on \mathbb{P}^2 . In other words, the rank of the divisor class group of S is at least two and if it is two, then the quartic curve R consists of four distinct lines on \mathbb{P}^2 .

Note that the point $\pi(P)$ must belong to the quartic curve R by Theorem 3.3 (1).

Suppose that the quartic curve R is smooth at $\pi(P)$. Let T_P be the unique curve in $|-K_S|$ that is singular at P . Then the log pair (S, T_P) is not log canonical at P by Theorem 3.3 (2).

The curve T_P consists of at most two irreducible components. Thus, if the rank of the divisor class group of S is at least 3, then $D \neq T_P$ by Lemma 3.5. If the rank of the divisor class group of S is two, then R is a union of four distinct lines. This implies that the support of T_P is an irreducible curve. Therefore, $D \neq T_P$ by Lemma 3.5.

Let μ be the greatest rational number such that $D' = (1 + \mu)D - \mu T_P$ is effective. Then (S, D') is not log canonical at P by Lemma 3.7. This contradicts Theorem 3.3 (2).

Thus, the point P must be a singular point of the surface S . Let $f: \bar{S} \rightarrow S$ be the blow up of the surface S at P . Then there exists a commutative diagram

$$\begin{array}{ccccc}
 & & W & & \\
 & \swarrow t & & \searrow f_2 & \\
 \bar{S} & & & & \mathbb{P}^1 \\
 & \searrow f & \swarrow f_1 & & \\
 & S & \xrightarrow{\phi} & \mathbb{P}^1 &
 \end{array}$$

where t is a birational morphism. Denote by E the f -exceptional curve and denote by \bar{D} the proper transform of the divisor D on the surface \bar{S} . The image of E_n by the birational morphism t is a point on the exceptional curve E . Denote this point by Q .

Note that the log pair $(\bar{S}, f^*(D))$ is not log canonical at Q .

Let C be the unique curve in the anticanonical linear system $|-K_S|$ whose proper transform by the blow up f passes through the point Q .

The curve C has at most two irreducible components. If the curve C is irreducible, then $D \neq C$ by Lemma 3.5. Suppose that the curve C has two irreducible components. If the rank of the divisor class group of S is greater than two, then $D \neq C$ by Lemma 3.5. If the rank of the divisor class group of S is two, then R is a union of four distinct lines, and hence (S, C) is log canonical. By Theorem 3.4, (S, D) must be log canonical as well. This is a contradiction.

Let μ be the greatest rational number such that $D' = (1 + \mu)D - \mu C$ is effective. The log pair (S, D') is not log canonical at P and $(\bar{S}, f^*(D'))$ is not log canonical at Q by Lemma 3.7. The same curve C is the curve in $|-K_S|$ whose proper transform by the blow up f passes through Q . By Theorem 3.4 the curve C is contained in the support of D' . This is a contradiction. \square

Remark 3.8. In order to prove Theorem 1.5 (I), we use Theorems 3.1, 3.3 and 3.4. In fact, they show that under the condition of Theorem 1.5 (I) the support of each tiger on S_d contains the support of an effective anticanonical divisor. Furthermore, on a del Pezzo surface S_d not listed in Theorem 1.5 (I) we can always find a tiger whose support does not contain the support of any effective anticanonical divisor. Indeed, using the effective anticanonical divisors listed in [23] and the cylinders constructed in the present article, we have hunted such tigers. Since this case-by-case hunting is a tedious job, we do not present such tigers here. We may rephrase Theorem 1.5 as follows:

Let S be a del Pezzo surface with at most du Val singularities. It has no $(-K_S)$ -cylinder if and only if the support of each tiger on S contains the support of an effective anticanonical divisor.

4. CONSTRUCTION OF CYLINDERS

In this section, we prove Theorem 1.5 (II). For a given singular del Pezzo surface S not listed in Theorem 1.5 (I) we find an effective anticanonical \mathbb{Q} -divisor D_S such that the complement of the support of D_S is isomorphic to $\mathbb{A}^1 \times Z$ for some smooth rational affine curve Z .

4.1. Construction for high degrees. We first construct $(-K_S)$ -polar cylinders for singular del Pezzo surfaces of degree 3 with only du Val singularities. Using this construction, we also obtain $(-K_S)$ -polar cylinders for singular del Pezzo surfaces of degrees ≥ 4 .

Theorem 4.1. *Let S be a singular cubic surface in \mathbb{P}^3 with only du Val singularities. Let P be a singular point of S and let L_1, \dots, L_r be the lines on S passing through P .*

- *There are positive rational numbers a_1, \dots, a_r such that the effective \mathbb{Q} -divisor*

$$a_1 L_1 + \dots + a_r L_r$$

is \mathbb{Q} -linearly equivalent to $-K_S$.

- If P is an ordinary double point, for a hyperplane section L of S such that it has a cuspidal point at P the set

$$S \setminus (L \cup L_1 \cup \dots \cup L_r)$$

is a cylinder.

- If P is a non-ordinary double point, then the set

$$S \setminus (L_1 \cup \dots \cup L_r)$$

is a cylinder.

In particular, S has a $(-K_S)$ -polar cylinder.

Proof. It is easy to see that there are lines L_1, \dots, L_r passing through P and $1 \leq r \leq 6$.

Let $\pi : S \dashrightarrow \mathbb{P}^2$ be the projection from P . It is a birational map and it contracts exactly the lines L_1, \dots, L_r . Let $f : \bar{S} \rightarrow S$ be the blow up at the point P and E be its exceptional divisor. Then the map $g := \pi \circ f : \bar{S} \rightarrow \mathbb{P}^2$ is defined everywhere. Let C be the image of E by g on \mathbb{P}^2 . It is a conic curve. Furthermore, it contains all the points $\pi(L_i)$. If P is an ordinary double point, then C is a smooth conic. If P is of type A_n , $n \geq 2$, then C consists of two distinct lines. If P is of type either D_n or E_n , then C is a (double) line.

Since the map π is defined by the linear system of hyperplane sections passing through P , the pull-back of $\frac{1}{2}C$ by π is an effective anticanonical \mathbb{Q} -divisor. Moreover, we immediately see

$$\pi^*\left(\frac{1}{2}C\right) = a_1 L_1 + \dots + a_r L_r$$

for some positive rational numbers a_i because the curve C passes through all the points $\pi(L_1), \dots, \pi(L_r)$.

It is easy to see

$$S \setminus (L_1 \cup \dots \cup L_r) \cong \bar{S} \setminus (E \cup \bar{L}_1 \cup \dots \cup \bar{L}_r) \cong \mathbb{P}^2 \setminus \text{Supp}(C),$$

where \bar{L}_i is the proper transform of L_i by f .

Suppose that P is a non-ordinary double point. Since the support of C consists of at most two lines, its complement is a cylinder.

Suppose that P is an ordinary double point. Then the conic C is smooth. Let L be a hyperplane section of S that has a cuspidal point at P . For a hyperplane section of S to have a cuspidal point, it has to be irreducible. Therefore, the hyperplane section L cannot meet L_i at a point other than P .

The proper transform of L by f is contained in the smooth locus of the surface \bar{S} and it is a smooth curve that meets the exceptional curve E tangentially at a single point. Note that it does not meet any of \bar{L}_i 's. Therefore, its image by g is a line tangent to the curve C . Note that the pull-back of a line tangent to C at a point other than $\pi(L_i)$ by the map π is such a hyperplane section as L . Consequently, the set

$$S \setminus (L \cup L_1 \cup \dots \cup L_r)$$

is a cylinder. This completes the proof. \square

Theorem 4.2. *Let S be a del Pezzo surface of degree $d \geq 4$ with at worst du Val singularities. Then it always contains a $(-K_S)$ -polar cylinder.*

Proof. Recall that we consider only del Pezzo surfaces of degrees ≤ 7 for the reason explained right after Corollary 1.6.

Let \tilde{S} be the minimal resolution of S and let A be a (-1) -curve on \tilde{S} . Let $\rho_1 : \Sigma_1 \rightarrow \tilde{S}$ be the blow up of \tilde{S} at a general point on A and let E_1 be its exceptional curve. Let $\rho_2 : \Sigma_2 \rightarrow \Sigma_1$ be the blow up of Σ_1 at a general point on E_1 and let E_2 be its exceptional curve. We repeat this procedure $(d - 3)$ times until we get a weak del Pezzo surface Σ_{d-3} of degree 3. Set

$\rho = \rho_{d-3} \circ \dots \circ \rho_1$ and denote the proper transforms of A and E_1, \dots, E_{d-4} by the same symbols. Contract all the (-2) -curves on Σ_{d-3} to get a del Pezzo surface Σ of degree 3 with du Val singularities. This contraction is denoted by $\psi : \Sigma_{d-3} \rightarrow \Sigma$.

The image of A by ψ is a singular point of Σ . We apply Theorem 4.1 to this singular point to get an effective anticanonical \mathbb{Q} -divisor D on Σ that defines a cylinder. This divisor contains all the lines passing through the point $\psi(A)$. Then

$$\psi^*(D) \sim_{\mathbb{Q}} -K_{\Sigma_{d-3}}.$$

We may write

$$\psi^*(D) = aA + \sum_{i=1}^{d-3} b_i E_i + \Delta,$$

where a and b_i 's are positive rational numbers and Δ is an effective \mathbb{Q} -divisor on Σ_{d-3} whose support contains none of A and E_i 's. Note that $\psi(E_{d-3})$ is a line passing through the point $\psi(A)$.

The divisor $aA + \rho(\Delta)$ on S is \mathbb{Q} -linearly equivalent to $-K_S$. Furthermore,

$$S \setminus \text{Supp}(aA + \rho(\Delta)) \cong \Sigma_{d-3} \setminus \text{Supp}\left(aA + \sum_{i=1}^{d-3} b_i E_i + \Delta\right) \cong S \setminus \text{Supp}(D).$$

Therefore, $aA + \rho(\Delta)$ defines a $(-K_S)$ -polar cylinder on S . \square

4.2. Construction for low degrees. Now we seek for an effective anticanonical \mathbb{Q} -divisor D_S that defines a cylinder on a given singular del Pezzo surface S of degree ≤ 2 not listed in Theorem 1.5 (I). To this end, instead of the singular surface S , we can consider its minimal resolution $f : \tilde{S} \rightarrow S$. Since we only allow du Val singularities on the surface S , the surface \tilde{S} is a smooth weak del Pezzo surface, i.e., a smooth surface with nef and big anticanonical class $-K_{\tilde{S}}$. On this smooth weak del Pezzo surface, it is enough to find an effective anticanonical \mathbb{Q} -divisor $D_{\tilde{S}}$ satisfying the following conditions:

- its support contains all the (-2) -curves on \tilde{S} ;
- the complement of the support of $D_{\tilde{S}}$ is isomorphic to $\mathbb{A}^1 \times Z$ for some smooth rational affine curve Z .

Then we can take the divisor D_S as $f(D_{\tilde{S}})$.

On the other hand, in order to find such a divisor $D_{\tilde{S}}$, we start with the projective plane \mathbb{P}^2 and one of the following effective anticanonical \mathbb{Q} -divisors $D_{\mathbb{P}^2}$ on \mathbb{P}^2 :

- a triple line $3L$;
- $a_1 L_1 + a_2 L_2$, where $a_1 + a_2 = 3$ and L_1, L_2 are distinct lines;
- $aL + bC$, where $a + 2b = 3$, C is an irreducible conic and L is a line tangent to the conic C ;
- $a_1 L_1 + a_2 L_2 + a_3 L_3$, where $a_1 + a_2 + a_3 = 3$ and L_1, L_2, L_3 are three distinct lines meeting at a single point.

Note that the complement $\mathbb{P}^2 \setminus \text{Supp}(D_{\mathbb{P}^2})$ is isomorphic to \mathbb{A}^2 , $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{\text{one point}\})$, $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{\text{one point}\})$, $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{\text{two points}\})$, respectively.

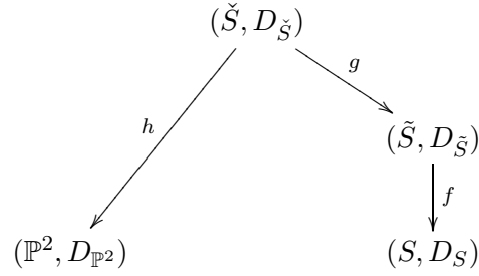
Let S be a given del Pezzo surface with du Val singularities and \tilde{S} be its minimal resolution. Starting from \mathbb{P}^2 with one of the divisors $D_{\mathbb{P}^2}$ we build a sequence of blow ups $h : \check{S} \rightarrow \mathbb{P}^2$ and a sequence of blow downs $g : \check{S} \rightarrow \tilde{S}$ with the following properties. Let $D_{\check{S}}$ be the log pull-back of $D_{\mathbb{P}^2}$ by h , i.e., the divisor such that

$$K_{\check{S}} + D_{\check{S}} = h^*(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}).$$

The divisor $D_{\check{S}}$ satisfies the following:

- (1) it is effective;
- (2) its support contains all the exceptional curves of h ;

(3) its support contains all the curves contracted by g .



Existence of such birational morphisms h and g shows that the given surface S admits a $(-K_S)$ -polar cylinder since

$$\mathbb{P}^2 \setminus \text{Supp}(D_{\mathbb{P}^2}) \cong \check{S} \setminus \text{Supp}(D_{\check{S}}) \cong \tilde{S} \setminus \text{Supp}(D_{\tilde{S}}) \cong S \setminus \text{Supp}(D_S),$$

where $D_{\tilde{S}} = g(D_{\check{S}})$ and $D_S = f(D_{\tilde{S}})$.

For a given del Pezzo surface of degree ≤ 2 with du Val singularities not listed in Theorem 1.5 (I), the method to construct such birational morphisms h and g is described in Tables 1 and 2 at the end.

4.3. The Table. For a given del Pezzo surface S of degree ≤ 2 with du Val singularities, in Tables 1 and 2, we provide the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g described in 4.2 in order to show how to construct a $(-K_S)$ -polar cylinder on S .

We read the tables in the following way.

In the first column the singularity types are given in normal size letters. The singularity types in small letters in Table 1 are those for del Pezzo surfaces of degree 2. These singularity types in small letters will be explained later.

The birational morphism h is obtained by successive blow ups with exceptional curves $E_{\textcircled{1}}, \dots, E_{\textcircled{n}}$ in this order. The configuration of these exceptional curves given in the third column shows how to take these blow ups. The exceptional curves $E_{\textcircled{1}}, \dots, E_{\textcircled{n}}$ are labelled by $\textcircled{1}, \dots, \textcircled{n}$, respectively, in the third column. The configuration in the third column also shows $D_{\mathbb{P}^2}$. We denote the proper transforms of lines from \mathbb{P}^2 by L_i (or L). We denote the proper transform of an irreducible conic from \mathbb{P}^2 by Q .

In the second column, the sum of the first divisor (tiger) and the second divisor (divisor contracted), if any, is the divisor $D_{\check{S}}$. If we have the second divisor in the second column, the birational morphism g is obtained by contracting curves drawn by dotted curves in the third column. The second divisor in the second column is contracted by g . Indeed, each component of the second divisor is depicted by a dotted curve in the third column. If we do not have the second divisor in the second column, then $\check{S} = \tilde{S}$ and the morphism g is the identity. The fat curves in the third column are the curves to be (-2) -curves on \tilde{S} . The thin lines with dots at one of the ends are the curves to be (-1) -curves on \tilde{S} . The wiggly lines are the curves to be non-negative curves on \tilde{S} .

In the second column, the curves without superscripts are (-2) -curves on \check{S} . The curves superscripted by black-circled numbers are the smooth rational curves on \check{S} with self-intersection numbers of the negatives of the black-circled numbers. The curves superscripted by the circled numbers are the smooth rational curves on \check{S} with self-intersection numbers of the circled numbers.

For a del Pezzo surface of degree 2 with a singularity type written in small letters in Table 1 the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g can be easily obtained by contracting one of the (-1) -curves (thin lines with dots at one of the ends) in the third column. Only for

singularity types D_4 , A_3 and A_2 they cannot be obtained in this way. For these three types, we provide the divisor $D_{\mathbb{P}^2}$ and the birational morphisms h and g in Table 2, separately.

The methods are given according to the singularity types of singular del Pezzo surfaces. Even though they show how to construct the birational morphisms h and g for *a seemingly single* del Pezzo surface S of a given singularity type, they indeed demonstrate how to obtain the birational morphisms h and g for *every* del Pezzo surface S of a given singularity type. Let us explain the reason.

Let S' be an arbitrary del Pezzo surface of a given singularity type and \tilde{S}' be its minimal resolution. The configurations of (-1) -curves and (-2) -curves on smooth weak del Pezzo surfaces are the same if the surfaces are of the same type. If the divisor $D_{\tilde{S}}$ in the table for the given singularity type consists of only negative curves, then we can immediately find a \mathbb{Q} -divisor $D_{\tilde{S}'}$ on the surface S' with the same configuration of the same kind of curves and the same coefficients. This is \mathbb{Q} -linearly equivalent to $-K_{\tilde{S}'}$. It is obvious that we can recover the birational morphisms h and g , in such a way that the divisor $D_{\tilde{S}'}$ plays the same role as $D_{\tilde{S}}$, by tracking back the blow downs and blow ups along the way given in the table for the given singularity type.

Now we consider the case when the divisor $D_{\tilde{S}}$ in the table for the given singularity type contains a non-negative curve. If we find a \mathbb{Q} -divisor $D_{\tilde{S}'}$ on the surface S' with the same configuration of the same kind of curves and the same coefficients, then the method presented in the table works for the surface S' , as in the previous case. To find such a \mathbb{Q} -divisor $D_{\tilde{S}'}$, we first notice from the table that the divisor $D_{\tilde{S}}$ contains at most one non-negative curve. Let F be the non-negative curve on \tilde{S} that appears in $D_{\tilde{S}}$ with coefficient $a > 0$. We have to show that such a non-negative curve always exists on the surface \tilde{S}' . To do so, put $D_{\tilde{S}}^0 = D_{\tilde{S}} - aF$. We can then find a \mathbb{Q} -divisor $D_{\tilde{S}'}^0$ on the surface S' with the same configuration of the same kind of curves and the same coefficients as $D_{\tilde{S}}^0$. Next we find a composition ψ of $9 - d$ blow downs starting from \tilde{S} to \mathbb{P}^2 . Let C_1, \dots, C_{9-d} be the negative curves contracted by the birational morphism ψ . We suppose that the first r curves C_1, \dots, C_r (possibly $r = 0$) intersect F and the others do not intersect F . We are then able to obtain the composition ψ' of the $9 - d$ blow downs starting from \tilde{S}' to \mathbb{P}^2 by contracting the negative curves C'_1, \dots, C'_{9-d} corresponding to the curves C_1, \dots, C_{9-d} , respectively, since the configurations of the negative curves on \tilde{S} and \tilde{S}' are the same. Then we see the divisor $\psi(D_{\tilde{S}})$ on \mathbb{P}^2 . The curve F is not contracted by ψ . Now we see that finding a \mathbb{Q} -divisor $D_{\tilde{S}'}$ on \tilde{S}' is equivalent to finding an irreducible curve F' of degree $\deg(\psi(F))$ on \mathbb{P}^2 such that

- $\psi'(D_{\tilde{S}'}^0) + aF'$ and $\psi(D_{\tilde{S}})$ have the same configuration;
- F' contains the points $\psi'(C'_1), \dots, \psi'(C'_r)$ but not the points $\psi'(C'_{r+1}), \dots, \psi'(C'_{9-d})$.

It is straightforward to find such an irreducible curve on \mathbb{P}^2 .

We can immediately find the negative curves for the morphisms ψ from the configurations in the third column for the singularity types with E_6 on del Pezzo surfaces of degree 1. For the singularity types with A_4 on del Pezzo surfaces of degree 1 we keep it in mind that there is always one (-1) -curve that meets the two (-2) -curves that are the ends of the chain of four (-2) -curves on \tilde{S} (see [21, Appendix]). For the singularity type A_2 on a del Pezzo surface of degree 2, we provide more detail in Example 4.3. This also helps us understanding how to use the tables.

Example 4.3. We explain how to construct a cylinder on a del Pezzo surface of degree 2 with singularity type A_2 .

On the projective plane \mathbb{P}^2 , take $D_{\mathbb{P}^2} = \frac{7}{4}L_1 + \frac{5}{4}L_2$, where L_1 and L_2 are distinct two lines. As shown in the third column for A_2 ($d = 2$), we take ten blow ups following the depicted instruction. Let $h : \tilde{S} \rightarrow \mathbb{P}^2$ be the composition of these ten blow ups. As explained at the

beginning of the section, $E_{\textcircled{1}}$ (resp. $E_{\textcircled{2}}, \dots, E_{\textcircled{10}}$) is the proper transform of the exceptional divisor of the first (resp. second, ..., tenth) blow up on the surface \check{S} . We then obtain

$$K_{\check{S}} + D_{\check{S}} = h^*(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}) \sim_{\mathbb{Q}} 0,$$

where

$$D_{\check{S}} = \left(\frac{3}{4}E_{\textcircled{1}} + \frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}} + \frac{5}{4}L_2 \right) + \left(\frac{6}{4}E_{\textcircled{2}} + \frac{5}{4}E_{\textcircled{3}} + \frac{7}{4}L_1 \right).$$

Here, the proper transforms of L_1 and L_2 by h are denoted by the same notation. The \mathbb{Q} -divisor $D_{\check{S}}$ is obtained by the sum of two \mathbb{Q} -divisors in the second column of the table. On the surface \check{S} , the curve L_2 is a (-5) -curve, the curve $E_{\textcircled{1}}$ is a (-3) -curve, the curves $E_{\textcircled{2}}, E_{\textcircled{3}}$ are (-2) -curves and the other eight curves in the second column of the table are (-1) -curves.

Starting from the (-1) -curve L_1 , we can contract $E_{\textcircled{2}}$ and $E_{\textcircled{3}}$ in turn to the smooth weak del Pezzo surface \tilde{S} corresponding to a del Pezzo surface S of degree 2 with singularity type A_2 . Denote the composition of these three blow downs by $g : \check{S} \rightarrow \tilde{S}$. Put

$$D_{\tilde{S}} = g \left(\frac{3}{4}E_{\textcircled{1}} + \frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}} + \frac{5}{4}L_2 \right).$$

This is an effective anticanonical \mathbb{Q} -divisor on the surface \tilde{S} .

Note that the curves $g(E_{\textcircled{1}})$ and $g(L_2)$ are the only (-2) -curves on the surface \tilde{S} and they intersect each other in the form of A_2 . Contracting these two (-2) -curves, we obtain a birational morphism $f : \tilde{S} \rightarrow S$, where S is a del Pezzo surface of degree 2 with one singular point of type A_2 . Put

$$D_S = f \circ g \left(\frac{1}{4}E_{\textcircled{4}} + \frac{1}{4}E_{\textcircled{5}} + \frac{1}{4}E_{\textcircled{6}} + \frac{1}{4}E_{\textcircled{7}} + \frac{1}{4}E_{\textcircled{8}} + \frac{1}{4}E_{\textcircled{9}} + \frac{1}{4}E_{\textcircled{10}} \right).$$

This is an effective anticanonical \mathbb{Q} -divisor on the surface S such that

$$S \setminus \text{Supp}(D_S) \cong \mathbb{P}^2 \setminus \text{Supp}(D_{\mathbb{P}^2}) \cong \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{\text{one point}\}).$$

Now we consider an arbitrary del Pezzo surface S' of degree 2 with one singular point of type A_2 . Let $f' : \tilde{S}' \rightarrow S'$ be the minimal resolution of the surface S' . The surface \tilde{S}' is a smooth weak del Pezzo surface of degree 2. Since it has the same configuration of negative curves as that of the weak del Pezzo surface \tilde{S} , we have the negative curves $E'_{\textcircled{1}}, E'_{\textcircled{2}}, E'_{\textcircled{3}}, E'_{\textcircled{5}}, \dots, E'_{\textcircled{10}}, L'_2$ on the surface \tilde{S}' corresponding to $g(E_{\textcircled{1}}), g(E_{\textcircled{2}}), g(E_{\textcircled{3}}), g(E_{\textcircled{5}}), \dots, g(E_{\textcircled{10}}), g(L_2)$, respectively, on the surface \tilde{S} . In order to construct a $(-K_{S'})$ -polar cylinder on the surface S' , it is enough to show that we can obtain the same kind irreducible curve $E'_{\textcircled{4}}$ on the surface \tilde{S}' as the 0-curve $g(E_{\textcircled{4}})$ on the surface \tilde{S} .

Let $\psi' : \tilde{S}' \rightarrow \mathbb{F}_2$ be the birational morphism obtained by contracting the six (-1) -curves $E'_{\textcircled{5}}, E'_{\textcircled{6}}, E'_{\textcircled{7}}, E'_{\textcircled{8}}, E'_{\textcircled{9}}, E'_{\textcircled{10}}$ to the Hirzebruch surface with (-2) -curve section. Instead of \mathbb{P}^2 we maps \tilde{S}' to \mathbb{F}_2 because this gives simpler explanation. However, its principle is the same. The image $\psi'(E'_{\textcircled{1}})$ is the negative section of \mathbb{F}_2 . The image $\psi'(L'_2)$ is irreducible and not contained in a fiber of $\mathbb{F}_2 \rightarrow \mathbb{P}^1$. The curve $\psi'(L'_2)$ intersects the section $\psi'(E'_{\textcircled{1}})$ at a single point.

We have the fiber of $\mathbb{F}_2 \rightarrow \mathbb{P}^1$ passing through the intersection point of $\psi'(L'_2)$ and $\psi'(E'_{\textcircled{1}})$. The proper transform of this fiber by ψ' will play the role of $E'_{\textcircled{4}}$. To be precise, denote the proper transform of the fiber by $E'_{\textcircled{4}}$. Then we put

$$D_{\tilde{S}'} = \frac{3}{4}E'_{\textcircled{1}} + \frac{1}{4}E'_{\textcircled{4}} + \frac{1}{4}E'_{\textcircled{5}} + \frac{1}{4}E'_{\textcircled{6}} + \frac{1}{4}E'_{\textcircled{7}} + \frac{1}{4}E'_{\textcircled{8}} + \frac{1}{4}E'_{\textcircled{9}} + \frac{1}{4}E'_{\textcircled{10}} + \frac{5}{4}L'_2.$$

This is an effective anticanonical \mathbb{Q} -divisor on the surface \tilde{S}' . We put

$$D_{S'} = f' \left(\frac{1}{4}E'_4 + \frac{1}{4}E'_5 + \frac{1}{4}E'_6 + \frac{1}{4}E'_7 + \frac{1}{4}E'_8 + \frac{1}{4}E'_9 + \frac{1}{4}E'_{10} \right).$$

This is an effective anticanonical \mathbb{Q} -divisor on the surface S' and we have

$$S' \setminus \text{Supp}(D_{S'}) \cong \tilde{S}' \setminus \text{Supp}(D_{\tilde{S}'}) \cong \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{\text{one point}\}).$$

Therefore, S' has a $(-K_{S'})$ -polar cylinder.

Remark 4.4. In fact, we have some freedom for the coefficients in the divisors $D_{\tilde{S}}$. We have fixed their coefficients simply to have better exposition in the table. For instance, let us reconsider Example 4.3. We here consider

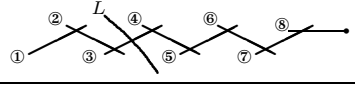
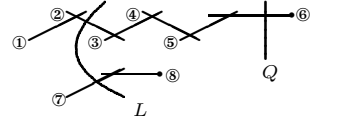
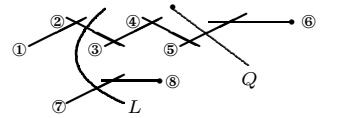
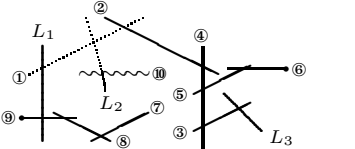
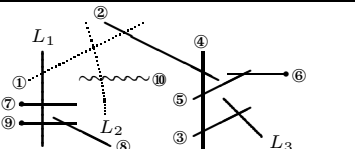
$$D_{\mathbb{P}^2} = (2 - \epsilon) L_1 + (1 + \epsilon) L_2$$

instead of $\frac{7}{4}L_1 + \frac{5}{4}L_2$. The proper transform of the divisor $D_{\mathbb{P}^2}$ by the birational morphism h is

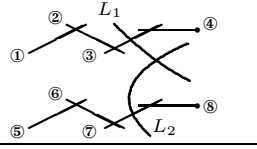
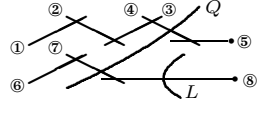
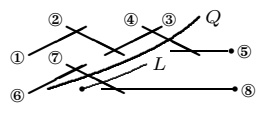
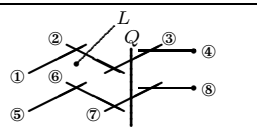
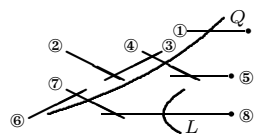
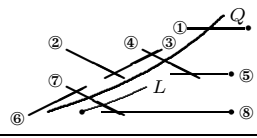
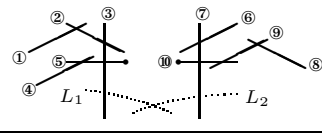
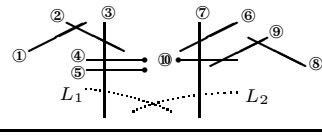
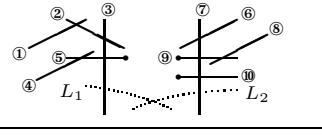
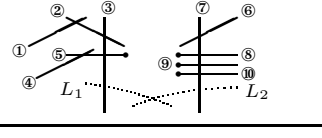
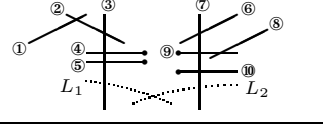
$$D_{\tilde{S}} = ((1 - \epsilon) E_1 + (1 - 3\epsilon) E_4 + \epsilon E_5 + \epsilon E_6 + \epsilon E_7 + \epsilon E_8 + \epsilon E_9 + \epsilon E_{10} + (1 + \epsilon) L_2) + ((2 - 2\epsilon) E_2 + (2 - 3\epsilon) E_3 + (2 - \epsilon) L_1).$$

For the divisor $D_{\tilde{S}}$ to be effective and to contain the exceptional divisors of the birational morphisms h and g , it is enough to take a rational number ϵ such that $0 < \epsilon < \frac{1}{3}$. In Example 4.3, we have simply chosen $\epsilon = \frac{1}{4}$. In almost all the other singularity types of the table, we may manipulate the coefficients in the divisors $D_{\tilde{S}}$ in the same way.

Table 1: Degree 1

Singularity Type	Tiger/ Divisor contracted (if any)	Construction
E_8 E_7	$2E_1 + 4E_2 + 6E_3 + 5E_4 + 4E_5 + 3E_6 + 2E_7 + E_8^\bullet + 3L$	
$E_7 + A_1$ $E_6, D_6 + A_1$	$\frac{5}{3}E_1 + \frac{10}{3}E_2 + \frac{8}{3}E_3 + 2E_4 + \frac{4}{3}E_5 + \frac{2}{3}E_6^\bullet + \frac{4}{3}E_7 + \frac{1}{3}E_8^\bullet + \frac{1}{3}Q + \frac{7}{3}L$	
E_7 E_6, D_6	$\frac{5}{3}E_1 + \frac{10}{3}E_2 + \frac{8}{3}E_3 + 2E_4 + \frac{4}{3}E_5 + \frac{1}{3}E_6^\bullet + \frac{4}{3}E_7 + \frac{1}{3}E_8^\bullet + \frac{1}{3}Q^\bullet + \frac{7}{3}L$	
$E_6 + A_2$ $D_5 + A_1, A_5 + A_2$	$\frac{12}{7}E_2^\bullet + \frac{10}{7}E_3 + \frac{15}{7}E_4 + \frac{8}{7}E_5 + \frac{1}{7}E_6^\bullet + \frac{1}{7}E_7 + \frac{2}{7}E_8 + \frac{3}{7}E_9^\bullet + \frac{1}{7}E_{10}^\bullet + \frac{8}{7}L_1^\bullet + \frac{5}{7}L_3$ $\frac{8}{7}L_2^\bullet + 2E_1$	
$E_6 + A_1$ $D_5 + A_1, D_5,$ $(A_5 + A_1)'$	$\frac{12}{7}E_2^\bullet + \frac{10}{7}E_3 + \frac{15}{7}E_4 + \frac{8}{7}E_5 + \frac{1}{7}E_6^\bullet + \frac{1}{7}E_7^\bullet + \frac{1}{7}E_8 + \frac{2}{7}E_9^\bullet + \frac{1}{7}E_{10}^\bullet + \frac{8}{7}L_1^\bullet + \frac{5}{7}L_3$ $\frac{8}{7}L_2^\bullet + 2E_1$	

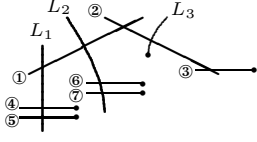
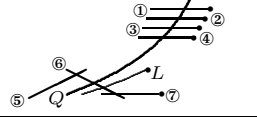
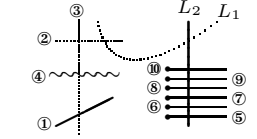
E_6 $D_5, (A_5)'$	$\frac{12}{7}E_2^\bullet + \frac{10}{7}E_3 + \frac{15}{7}E_4 + \frac{8}{7}E_5 + \frac{1}{7}E_6^\bullet + \frac{1}{7}E_7^\bullet + \frac{1}{7}E_8^\bullet + \frac{1}{7}E_9^\bullet + \frac{1}{7}E_{10}^\bullet + \frac{8}{7}L_1^\bullet + \frac{5}{7}L_3$ $\frac{8}{7}L_2^\bullet + 2E_1$	
D_8 $D_6 + A_1, A_7$	$\frac{3}{4}E_1 + \frac{3}{2}E_2 + \frac{7}{4}E_3 + 2E_4 + \frac{9}{4}E_5 + \frac{5}{2}E_6 + \frac{3}{2}E_7 + \frac{1}{2}E_8^\bullet + \frac{1}{2}L^\bullet + \frac{5}{4}Q$	
D_7 $D_5 + A_1, A_6$	$\frac{3}{4}E_1 + \frac{3}{2}E_2 + \frac{7}{4}E_3 + 2E_4 + \frac{9}{4}E_5 + \frac{5}{2}E_6 + \frac{1}{4}E_7^\bullet + \frac{1}{4}E_8^\bullet + \frac{1}{2}L + \frac{5}{4}Q$	
$D_6 + 2A_1$ $D_4 + 3A_1,$ $(A_5 + A_1)''$	$2E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5 + \frac{2}{5}E_6^\bullet + \frac{1}{5}E_7 + \frac{2}{5}E_8^\bullet + \frac{6}{5}L_1 + \frac{6}{5}L_2 + \frac{3}{5}L_3$	
$D_6 + A_1$ $D_4 + 2A_1, (A_5)''',$ $(A_5 + A_1)''$	$2E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5 + \frac{2}{5}E_6^\bullet + \frac{1}{5}E_7^\bullet + \frac{1}{5}E_8^\bullet + \frac{6}{5}L_1 + \frac{6}{5}L_2 + \frac{3}{5}L_3$	
D_6 $D_4 + A_1, (A_5)''$	$2E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5^\bullet + \frac{1}{5}E_6^\bullet + \frac{1}{5}E_7^\bullet + \frac{1}{5}E_8^\bullet + \frac{6}{5}L_1 + \frac{6}{5}L_2 + \frac{3}{5}L_3$	
$D_5 + A_3$ $2A_3 + A_1,$ $A_4 + A_2$	$\frac{4}{5}E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5 + \frac{2}{5}E_6 + \frac{3}{5}E_7 + \frac{4}{5}E_8^\bullet + \frac{3}{5}L + \frac{6}{5}Q$	
$D_5 + A_2$ $A_3 + A_2 + A_1,$ $A_4 + A_2, A_4 + A_1$	$\frac{4}{5}E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5^\bullet + \frac{1}{5}E_6 + \frac{2}{5}E_7 + \frac{3}{5}E_8^\bullet + \frac{3}{5}L + \frac{6}{5}Q$	
$D_5 + 2A_1$ $A_3 + 3A_1, A_4 + A_1$	$\frac{4}{5}E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5 + \frac{2}{5}E_6^\bullet + \frac{1}{5}E_7 + \frac{2}{5}E_8^\bullet + \frac{3}{5}L + \frac{6}{5}Q$	
$D_5 + A_1$ $(A_3 + 2A_1)',$ $A_4 + A_1, A_4$	$\frac{4}{5}E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5^\bullet + \frac{1}{5}E_6^\bullet + \frac{1}{5}E_7 + \frac{2}{5}E_8^\bullet + \frac{3}{5}L + \frac{6}{5}Q$	
D_5 $(A_3 + A_1)', A_4$	$\frac{4}{5}E_1 + \frac{8}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4^\bullet + \frac{1}{5}E_5^\bullet + \frac{1}{5}E_6^\bullet + \frac{1}{5}E_7^\bullet + \frac{1}{5}E_8^\bullet + \frac{3}{5}L + \frac{6}{5}Q$	
$2D_4, D_4 + A_3$ D_4	$D_4 + A_2, D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1,$	No Cylinder

A_8 $A_5 + A_2$	$\frac{1}{2}E_1 + E_2 + \frac{3}{2}E_3 + \frac{1}{2}E_4^\bullet + \frac{1}{2}E_5 + E_6 + \frac{3}{2}E_7 + \frac{1}{2}E_8^\bullet + \frac{3}{2}L_1 + \frac{4}{2}L_2$	
$A_7 + A_1$ $2A_3 + A_1,$ $(A_5 + A_1)''$	$\frac{1}{3}E_1 + \frac{2}{3}E_2 + E_3 + \frac{4}{3}E_4 + \frac{1}{3}E_5^\bullet + \frac{2}{3}E_6 + \frac{4}{3}E_7 + \frac{2}{3}E_8^\bullet + \frac{1}{3}L + \frac{4}{3}Q$	
$(A_7)'$ $2A_3, (A_5 + A_1)''$	$\frac{1}{3}E_1 + \frac{2}{3}E_2 + E_3 + \frac{4}{3}E_4 + \frac{1}{3}E_5^\bullet + \frac{2}{3}E_6 + \frac{4}{3}E_7 + \frac{1}{3}E_8^\bullet + \frac{1}{3}L^\bullet + \frac{4}{3}Q$	
$(A_7)''$ $A_4 + A_2,$ $(A_5 + A_1)'$	$\frac{7}{12}E_1 + \frac{7}{6}E_2 + \frac{19}{12}E_3 + \frac{7}{12}E_4^\bullet + \frac{5}{12}E_5 + \frac{5}{6}E_6 + \frac{5}{4}E_7 + \frac{1}{4}E_8^\bullet + \frac{1}{6}L^\bullet + \frac{17}{12}Q$	
$A_6 + A_1$ $A_3 + A_2 + A_1,$ $A_4 + A_1$	$\frac{5}{12}E_1^\bullet + \frac{5}{12}E_2 + \frac{5}{6}E_3 + \frac{5}{4}E_4 + \frac{1}{4}E_5^\bullet + \frac{7}{12}E_6 + \frac{7}{6}E_7 + \frac{1}{3}E_8^\bullet + \frac{1}{6}L + \frac{17}{12}Q$	
A_6 $A_3 + A_2, A_4 + A_1$	$\frac{5}{12}E_1^\bullet + \frac{5}{12}E_2 + \frac{5}{6}E_3 + \frac{5}{4}E_4 + \frac{1}{4}E_5^\bullet + \frac{7}{12}E_6 + \frac{7}{6}E_7 + \frac{1}{6}E_8^\bullet + \frac{1}{6}L^\bullet + \frac{17}{12}Q$	
$A_5 + A_2 + A_1$ $3A_2, A_3 + 3A_1$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3^\bullet + \frac{1}{5}E_4 + \frac{2}{5}E_5^\bullet + \frac{3}{5}E_6 + \frac{6}{5}E_7^\bullet + \frac{1}{5}E_8 + \frac{2}{5}E_9 + \frac{3}{5}E_{10}^\bullet$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^\bullet$	
$A_5 + A_2$ $3A_2, A_3 + 3A_1$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3^\bullet + \frac{1}{5}E_4 + \frac{2}{5}E_5^\bullet + \frac{3}{5}E_6 + \frac{6}{5}E_7^\bullet + \frac{1}{5}E_8 + \frac{2}{5}E_9 + \frac{3}{5}E_{10}^\bullet$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^\bullet$	
$A_5 + 2A_1$ $2A_2 + A_1, A_3 + 3A_1,$ $(A_3 + 2A_1)''$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3^\bullet + \frac{1}{5}E_4 + \frac{2}{5}E_5^\bullet + \frac{3}{5}E_6 + \frac{6}{5}E_7^\bullet + \frac{1}{5}E_8 + \frac{2}{5}E_9 + \frac{1}{5}E_{10}^\bullet$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^\bullet$	
$(A_5 + A_1)'$ $(A_3 + 2A_1)'', 2A_2$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3^\bullet + \frac{1}{5}E_4 + \frac{2}{5}E_5^\bullet + \frac{3}{5}E_6 + \frac{6}{5}E_7^\bullet + \frac{1}{5}E_8^\bullet + \frac{1}{5}E_9^\bullet + \frac{1}{5}E_{10}^\bullet$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^\bullet$	
$(A_5 + A_1)''$ $(A_3 + A_1)'',$ $2A_2 + A_1$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3^\bullet + \frac{1}{5}E_4 + \frac{2}{5}E_5^\bullet + \frac{3}{5}E_6 + \frac{6}{5}E_7^\bullet + \frac{1}{5}E_8 + \frac{2}{5}E_9^\bullet + \frac{1}{5}E_{10}^\bullet$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^\bullet$	

A_5 $(A_3 + A_1)''$, $2A_2$	$\frac{2}{5}E_1 + \frac{4}{5}E_2 + \frac{6}{5}E_3 + \frac{1}{5}E_4 + \frac{1}{5}E_5 + \frac{3}{5}E_6 + \frac{6}{5}E_7 + \frac{1}{5}E_8 + \frac{1}{5}E_9 + \frac{1}{5}E_{10}$ $\frac{7}{5}L_1 + \frac{8}{5}L_2^{\bullet}$	
$2A_4$ $A_3 + A_2 + A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{5}E_{10} + \frac{3}{10}E_{11} + \frac{2}{5}E_{12} + \frac{1}{2}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
$A_4 + A_3$ $A_3 + A_2 + A_1$, $2A_2 + A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{10}E_{10} + \frac{1}{5}E_{11} + \frac{3}{10}E_{12} + \frac{2}{5}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
$A_4 + A_2 + A_1$ $A_2 + 3A_1$, $2A_2 + A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{5}E_{10} + \frac{3}{10}E_{11} + \frac{1}{10}E_{12} + \frac{1}{5}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
$A_4 + A_2$ $2A_2 + A_1$, $A_2 + 2A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{10}E_{10} + \frac{1}{10}E_{11} + \frac{1}{5}E_{12} + \frac{3}{10}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
$A_4 + 2A_1$ $A_2 + 3A_1$, $A_2 + 2A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{10}E_{10} + \frac{1}{5}E_{11} + \frac{1}{10}E_{12} + \frac{1}{5}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
$A_4 + A_1$ $A_2 + 2A_1$, $A_2 + A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{10}E_{10} + \frac{1}{10}E_{11} + \frac{1}{10}E_{12} + \frac{1}{5}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	
A_4 $A_2 + A_1$	$\frac{19}{30}E_1 + \frac{1}{6}E_5 + \frac{11}{30}E_6 + \frac{11}{15}E_7 + \frac{11}{10}E_8 + \frac{1}{10}E_9 + \frac{1}{10}E_{10} + \frac{1}{10}E_{11} + \frac{1}{10}E_{12} + \frac{1}{10}E_{13}$ $\frac{19}{15}E_2 + \frac{9}{10}E_3 + \frac{7}{6}E_4 + \frac{49}{30}L_1^{\bullet} + \frac{41}{30}L_2$	

$2A_3 + 2A_1, 2A_3 + A_1, (2A_3)', (2A_3)'', A_3 + A_2 + 2A_1,$ $A_3 + A_2 + A_1, A_3 + A_2, A_3 + 4A_1, A_3 + 3A_1, (A_3 +$ $2A_1)', (A_3 + 2A_1)'', A_3 + A_1, A_3, 4A_2, 3A_2 + A_1,$ $3A_2, 2A_2 + 2A_1, 2A_2 + A_1, 2A_2, A_2 + 4A_1, A_2 + 3A_1,$ $A_2 + 2A_1, A_2 + A_1, A_2, 6A_1, 5A_1, (4A_1)', (4A_1)'',$ $3A_1, 2A_1, A_1$	No Cylinder
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Table 2: Degree 2

Singularity Type	Tiger/ Divisor contracted (if any)	Construction
D_4	$2E_1 + \frac{4}{3}E_2 + \frac{1}{3}E_3 + \frac{1}{3}E_4 + \frac{1}{3}E_5 + \frac{1}{3}E_6 +$ $\frac{1}{3}E_7 + \frac{4}{3}L_1 + \frac{4}{3}L_2 + \frac{1}{3}L_3$	
A_3	$\frac{1}{4}E_1 + \frac{1}{4}E_2 + \frac{1}{4}E_3 + \frac{1}{4}E_4 + \frac{3}{4}E_5 + \frac{3}{2}E_6 +$ $\frac{1}{2}E_7 + \frac{1}{2}L + \frac{5}{4}Q$	
A_2	$\frac{3}{4}E_1 + \frac{1}{4}E_4 + \frac{1}{4}E_5 + \frac{1}{4}E_6 + \frac{1}{4}E_7 + \frac{1}{4}E_8 +$ $\frac{1}{4}E_9 + \frac{1}{4}E_{10} + \frac{5}{4}L_2$ $\frac{6}{4}E_2 + \frac{5}{4}E_3 + \frac{7}{4}L_1$	
$6A_1, 5A_1, (4A_1)', (4A_1)'', (3A_1)', (3A_1)'', 2A_1, A_1$		No Cylinder

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